Exact solutions of some urn models of relaxation in glassy dynamics

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We consider two simple models, called ''urn models,'' for a general N-ball, M-urn problem. These models find applications in the study of relaxation in glassy dynamics. We obtain exact analytical results in these two cases for the average relaxation time τ to reach the ground state. In model I we also obtain the functional dependance of τ for large N, and in model II we obtain an asymptotic $(N\to\infty)$ dependence of τ as a function of the number of urns M. [S1063-651X(99)03207-9]

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Urn model problems are classical problems in probability theory [1,2] which have been widely studied. An attractive feature of these problems is that they are easy to formulate but not always easy to solve. The solutions obtained have, therefore, sometimes led to new mathematical techniques and insights. These problems have also been popular with physicists as models of physical processes. In the last few years, many papers [3–7] have appeared in literature on some specific urn models as models of the slow dynamics of glassy systems. Objections may be raised on the grounds that the models are far too simplistic to really capture all the features of the complex system. We feel that the study of these models is interesting in its own right, and does show some features such as a critical slowing down as the system approaches the ground state. In this paper, we obtain exact analytical solutions for two urn models. Ritort's model [4] is probably the first such model used for this purpose. In this model one considers N distinguishable balls which are distributed in M boxes. The dynamics is defined as follows. At each time step a ball is picked up randomly and independently, and moved to a random nonempty box. Moving the ball to an empty box is not allowed. This restriction constitutes what is popularly known as the entropy barrier, since, as the number of empty boxes increases with time, there are fewer and fewer boxes available for the transference of the ball, i.e., fewer paths in the ''phase space'' where energy can decrease. (Energy is defined as minus the number of empty boxes present at any time.) Thus relaxation of energy becomes slower with time. Ritort's model showed that the relaxation time τ averaged over an ensemble of initial configurations of N balls in M boxes to the ground state goes asymptotically as

$$\tau = 0.39\exp(0.67N). \tag{1}$$

Making use of the fact that in Ritort's urn model, the system, before it reaches the ground state (i.e., all the balls are in one box), has to pass essentially through a specific configuration where one ball is in one urn and $N-1$ balls are in another urn, Lipovski [5] found an exact expression for τ for the N-ball and two-urn model problem, which enabled him to find a lower bound for τ for the N-ball M-urn problem as $\tau = 2^{(N-1)}+N-3$ by a recursive analysis. Complementing upon Lipowski's work, Murthy and Kehr [6] considered the two-urn problem and solved it for an arbitrary initial distribution of N balls [i.e., any K balls in one urn and the remaining $(N$-$K)$ balls in another urn], and found upper and lower bounds for the relaxation time for various initial arbitrary states of the system size (N). They concluded that $K=1$ sets the principal time scale of the problem, and the relaxation from other states takes negligibly more time than this for large systems. To the best of our knowledge there exists no exact analytical solution of this problem for $M(>2)$-urn N-ball case. To this end, in this paper we consider two such models, namely, models I and II (to be defined explicitly below), and find an exact analytical formula for the relaxation time τ for these two models. It may be noted that our model I is the same as model C of Ref. [7], whereas our model II differs from their model A.

Model I: We start with an initial distribution of N balls in M urns given by $\overline{K}_0=(k_0^1,k_0^2,\ldots,k_0^M)$, where k_0^i is the number of balls in the ith urn at the start. At each time step two balls are picked out of N balls such that every pair has an equal probability of being picked up. Both balls are then placed into the urn to which the second ball belonged. The dynamics is such that the ground state corresponds to all coming into a single urn. The quantity of interest is the average number of time steps τ required for this event to occur for the first time. This is given by

$$\tau = \sum_{i=1}^{M}\sum_{n=0}^{\infty} n f(k_i,n), \tag{2}$$

with

$$\sum_{n=0}^{\infty} f(k_i,n) = f_0(k_i), \tag{3}$$

where $f(k_i,n)$ is the probability that the ith urn contains all N balls for the first time after n time steps, $f_0(k_i)$ is the probability that the ith urn is filled with all N balls, and $f(k_i,n)/f_0(k_i)$ is the conditional probability that this event occurs for the first time at time n. Notice that the dynamics of the process is such that if any urn becomes empty, it automatically goes out of the reckoning. Thus our model is

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exactly like model C of Ref. [7]. Though this problem can be solved directly, we found it convenient to solve it in two stages. In the first stage, the average time taken for a specified urn to either contain all N balls or become completely empty was computed, and this was then used to compute the quantity of interest.

Let $g(k,m)$ be the probability that an urn contains all N balls or is completely empty for the first time after m time steps given that it had k balls at the start (here $k=k_0^i$; for convenience we have dropped the superscript and subscript). We note that the dynamics of the process is such that the distribution of the balls in the other $(M-1)$ urns does not affect this quantity. We then write the recursion relation

$$g(k,m+1)=\frac{k(N-k)}{N(N-1)}[g(k+1,m)+g(k-1,m)]+\left(1-\frac{2k(N-k)}{N(N-1)}\right)g(k,m). \tag{4}$$

Multiplying by m and summing, we obtain

$$m_k=0.5[m_{k+1}+m_{k-1}]+\frac{N(N-1)}{2k(N-k)}, \tag{5}$$

where $m_k=\Sigma_{m=0}^{\infty}mg(k,m)$, and Eq. (4) is to be solved with the boundary conditions

$$m_0=0; \quad m_N=0. \tag{6}$$

Defining $\Delta_k=m_k-m_{k+1}$ we have from Eq. (5),

$$\Delta_k=\Delta_{k-1}+\frac{N(N-1)}{k(N-k)}. \tag{7}$$

Here the method of solution differs slightly for N even and N odd. We solve it for the case $N=2L$. Using Eq. (7), we have

$$\Delta_L=\Delta_{L-1}+\frac{2L(2L-1)}{L^2}. \tag{8}$$

However, because of symmetry, $\Delta_{L-1}=-\Delta_L$. Therefore,

$$\Delta_L=\frac{2L-1}{L}. \tag{9}$$

Using Eq. (7) repeatedly, and summing, we have

$$\Delta_i=\Delta_L+N(N-1)\sum_{k=L+1}^{i}\frac{1}{k(N-k)} \tag{10}$$

and

$$m_{N-1}=\Delta_{N-1}=\frac{2L-1}{L}+\sum_{k=L+1}^{N-1}\frac{N(N-1)}{k(N-k)}. \tag{11}$$

We also obtain

$$m_i=\sum_{j=i}^{N-1}\Delta_j=\frac{(N-i)(2L-1)}{L}+\sum_{j=i}^{N-1}\sum_{k=L+1}^{j}\frac{N(N-1)}{k(N-k)}$$
$$=\frac{(N-i)(2L-1)}{L}+\sum_{k=L+1}^{N-1}\frac{N(N-1)}{k}. \tag{12}$$

Now, in order to have all the balls in one urn we use the quantity $f_0(k)$, which is the probability that an urn containing k balls at the start contains all N balls at any time. Also let $f(k,n)/f_0(k)$ be the conditional probability that this urn contains all the N balls for the first time at time n:

$$f_0(k)=\sum_{n=0}^{\infty}f(k,n). \tag{13}$$

The starting equation for $f(k,n)$ is the same as in the earlier case:

$$f(k,n+1)=\frac{k(N-k)}{N(N-1)}[f(k+1,n)+f(k-1,n)]+\left(1-\frac{2k(N-k)}{N(N-1)}\right)f(k,n). \tag{14}$$

Summing Eq. (14) over all n, we obtain the recursion relation for $f_0(k)$ as

$$f_0(k)=0.5[f_0(k+1)+f_0(k-1)] \qquad \text{for} \qquad 1\leqslant k\leqslant N-1. \tag{15}$$

This is to be solved subject to the boundary conditions

$$f_0(N)=1,$$
$$f_0(0)=0. \tag{16}$$

Equations (15) and (16) yield

$$f_0(k)=\frac{k}{N}. \tag{17}$$

Now we use this quantity $f_0(k)$ to find the number of time steps required to obtain all the balls in one urn. We multiply Eq. (14) by $n+1$, and sum over all n, to obtain

$$n_k=0.5[n_{k+1}+n_{k-1}]+0.5[f_0(k+1)+f_0(k-1)]+\frac{N(N-1)-2k(N-k)}{2k(N-k)}f_0(k). \tag{18}$$

The boundary conditions are again $n_N=0$ and $n_0=0$. Defining $\Delta_k=n_k-n_{k+1}$, we have

$$\Delta_k=\Delta_{k-1}+g_k, \tag{19}$$

where $g_k=f_0(k+1)+f_0(k-1)+([N(N-1)-2k(N-k)]/2k(N-k))f_0(k)$. Substituting for $f_0(k)$ from Eq. (17), and simplifying, we have

$$g_k=\frac{N-1}{N-k}. \tag{20}$$

This yields

$$\Delta_i = \Delta_{N-1} - G_i$$

where

$$G_i = \sum_{k=i+1}^{N-1} g_k . \tag{21}$$

Therefore,

$$\begin{aligned} n_L &= \sum_{j=L}^{N-1} \Delta_j \\ &= (N-L)\Delta_{N-1} - \sum_{j=L}^{N-1} \sum_{k=j+1}^{N-1} g_k \\ &= (N-L)\Delta_{N-1} - \sum_{k=L+1}^{N-1} g_k \sum_{j=L}^{k-1} 1 \\ &= (N-L)\Delta_{N-1} - \sum_{k=L+1}^{N-1} \frac{(2k-N)(N-1)}{2(N-k)}. \end{aligned} \tag{22}$$

Because of symmetry, we have

$$n_L = (1/2) m_L . \tag{23}$$

Substituting for $'n_L$ from Eq. (12), we have

$$n_L = \frac{2L-1}{2} + \sum_{k=L+1}^{2L-1} \frac{N(N-1)}{2k} \tag{24}$$

for large N; this yields the result $n_L \sim N^2/2ln2$.

We may rewrite Eq. (22) as

$$n_{N-1} = \frac{n_L + \sum_{L+1}^{N-1} \frac{(2k-N)(N-1)}{2(N-k)}}{N-L}. \tag{25}$$

With a little algebra we obtain, for large N, $n_{N-1} \sim N \ln N - N$, and using this result and Eq. (2) for large N we obtain

$$\tau \sim N^2. \tag{26}$$

The functional dependance of τ on the number of urns M will depend on the initial distribution $\overline{K_0}$ of the balls in the urns.

Model II: In this model at each time step one ball out of the N balls initially distributed arbitrarily in M urns is drawn randomly and placed randomly in an urn different from the one from which it was drawn. The quantity we are interested in is the time taken for all the balls to move into a single urn for the first time. In the case considered by us, even if an urn becomes empty at some time, it is still taken into consideration, unlike in model A of Ref. [7], where any urn, when it becomes empty, is taken out of the reckoning. We define two events. Event 1: Urn i is filled for the first time assuming, that the dynamics remains the same till this event. Event 2: The first time any of the M urns is filled. Event 1: Let $f(k_i, n+1)$ be the probability that urn i is filled for the first time at time $n+1$, given that at time 0 it had k_i balls. Note that this probability is independent of the distribution of the remaining $N-k_i$ balls in the other $M-1$ urns. Then $f(k_i, n+1)$, satisfies the recurrence relation

$$f(k,n+1) = \frac{k}{N} f(k-1,n) + \frac{N-k}{N(M-1)} f(k+1,n) + \frac{(N-k)(M-2)}{N(M-1)} f(k,n) \tag{27}$$

(here we have dropped the subscript i of k for ease), which yields

$$n_k = \frac{k}{N} n_{k-1} + \frac{N-k}{N(M-1)} n_{k+1} + \frac{(N-k)(M-2)}{N(M-1)} n_k + 1. \tag{28}$$

This gives

$$\frac{N-k}{N(M-1)} \Delta_k = \frac{k}{N} \Delta_{k-1} + 1,$$

$$\Delta_0 = M-1. \tag{29}$$

After some algebra we obtain

$$\Delta_k = S_{1k} \Delta_0 + \sum_{j=1}^{k} t_j S_{j+1,k} , \tag{30}$$

where

$$S_{jk} = \prod_{l=j}^{k} s_l ,$$

$$s_l = \frac{(M-1)l}{N-l}, \tag{31}$$

and

$$t_j = \frac{N(M-1)}{N-j}.$$

We also have $n_N = 0$. Thus

$$n_i = \sum_{k=i}^{N-1} \Delta_k . \tag{32}$$

Event 2: Let us consider the case of M urns. Let $\vec{K} = k_1, k_2, \ldots, k_M$ and $g_i(\vec{K}, n)$ be the probability that urn i is filled up at time n before any of the other urns gets filled up:

$$f(k_i, n) = g_i(\vec{K}, n) + \sum_{j \neg i}^{M} \sum_{n'=0}^{n} g_j(\vec{K}, n') f(0, n-n') . \tag{33}$$

Note that g_i depends on $\overline{K_0}$ the full initial distribution. The moment generating function of Eq. (33) yields

$$\tilde{f}(k_i, z) = \tilde{g}_i(\vec{K}, z) + \sum_{j \neg i}^{M} \tilde{g}_j(\vec{K}, z) \tilde{f}(0, z), \tag{34}$$

where

$$\tilde{f}(k_i,z)=\sum_{n=0}^{\infty} f(k_i,n)z^n. \tag{35}$$

Summing over all i $(i=1,2,\ldots,M)$ and defining $G(\vec{K},z)$ as

$$G(\vec{K},z)=\sum_{i=1}^{M} \tilde{g}_i(\vec{K},z), \tag{36}$$

we obtain

$$G(\vec{K},z)=\frac{\sum_{i=1}^{M} \tilde{f}(k_i,z)}{1+(M-1)\tilde{f}(0,z)}. \tag{37}$$

Therefore, $\tau(\overrightarrow{K_0})$, the mean time required for all the balls to come into one urn for the first time, is given by

$$\tau(\overrightarrow{K_0})=\left.\frac{dG}{dz}\right|_{z=1}=\frac{\sum_{i=1}^{M} n_{k_i}}{M}-\frac{M-1}{M}n_0. \tag{38}$$

We can now find an asymptotic $(N\to\infty)$ expression for $\tau(\overrightarrow{K_0})$. From Eq. (30), after a little algebra, it is easy to see that $\Delta_{N-1}=M^N-1$, which implies $n_{N-1}=n_N+\Delta_{N-1}\sim M^N$. Further, $\Delta_{N-2}\sim M^N/N\ll\Delta^{N-1}$. Similarly, the other differences are also much smaller than Δ_{N-1}. Therefore, as a first approximation for large N we can take all the n_{k_i}'s to be equal to M^N. Substitution of these n_{k_i}'s in Eq. (38) gives $\tau(\overrightarrow{K_0})\approx M^{N-1}$ (for $N\to\infty$).

Thus, to conclude, we see that the asymptotic relaxation time τ crucially depends upon the particular model one chooses; it varies as N^2 with model I and as M^{N-1} with model II. It would be interesting to obtain the exact analytical solution of the N-ball M-urn ''pure entropic barrier problem'' of Ritort [4], wherein the urn which becomes empty is taken out of the process.

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